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CONFLICT AMONG TESTING PROCEDURES?(U)
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1. Introduction

Savin [1976] and Berndt and Savin [1977], among others, have pointed out that an inequality relation exists between the Lagrange Multiplier Test (LM), the Wald Test (W), and the Likelihood Ratio Test (LR). However, since all tests converge to the same limiting Chi-square distribution they are usually compared against the same critical value. This raises the possibility of conflicting conclusions from the three tests.

Kohler [1979] and Vandaele [1981] have shown that the three tests are monotonic functions of each other. This implies that they have identical power characteristics. In particular, if the probabilities of Type I errors are equal among the three tests, they have to have the same probability of Type II errors as well. In essence we are dealing with one and the same test.

In this paper we review briefly how the tests are related and why the inequality relation exists. We then derive criteria which allow us to determine which test is more appropriate in a given situation. This should resolve possible conflicts for at least some sets of circumstances.

2. The Three Tests and the Inequality Among Them

Consider the model:

$$Y = X\beta + \varepsilon \tag{1}$$

$$\varepsilon \sim N(0,\sigma^2 I) \quad . \tag{2}$$

Let $\hat{\beta}$ and $\hat{\sigma}^2$ be the maximum-likelihood estimators obtained by unconstrained maximization of the likelihood function, and $\tilde{\beta}$ and $\tilde{\sigma}^2$ the corresponding estimators obtained by maximizing the likelihood function subject to the constraint $R\tilde{\beta} = r$. Furthermore, we shall need an estimator for the Lagrange multiplier ($\hat{\mu}$) and the ratio of the constrained to unconstrained maxima of the likelihood function (λ). The three test statistics can be written as:

$$LR = -2 \log \lambda \tag{3}$$

$$W = [(r - R\hat{\beta})'(R(X'X)^{-1}R')^{-1}(r - R\hat{\beta})]/\hat{\sigma}^2 \tag{4}$$

$$LM = [\hat{\mu}(R(X'X)^{-1}R')\hat{\mu}]/\tilde{\sigma}^2 \quad . \tag{5}$$

To simplify the notation let $A = R(\frac{1}{n} X'X)^{-1}R'$. From the first-order conditions for maximizing the likelihood function subject to the constraint we can obtain an expression for $\hat{\mu}$:

$$\hat{\mu} = (R(X'X)^{-1}R')^{-1}(r - R\hat{\beta}) \quad . \tag{6}$$

We can now rewrite Eqs. (4) and (5) as

(7) $$W = n[(r - R\hat{\beta})'A^{-1}(r - R\hat{\beta})]/\hat{\sigma}^2$$

(8) $$LM = n[(r - R\hat{\beta})'A^{-1}(r - R\hat{\beta})]/\tilde{\sigma}^2 \quad .$$

Recall that $\tilde{\sigma}^2 = \hat{\sigma}^2 + (r - R\hat{\beta})'A^{-1}(r - R\hat{\beta})$ and $\lambda = (\tilde{\sigma}^2/\hat{\sigma}^2)^{-n/2}$. Adopting the shorthand notation θ for $\tilde{\sigma}^2/\hat{\sigma}^2$ we can rewrite the three tests as:

(9) $$LR = n \cdot \log(\theta)$$

(10) $$W = n \cdot (\theta - 1)$$

(11) $$LM = n \cdot (1 - \frac{1}{\theta}) \quad .$$

Given that $\theta \geq 1$ with probability 1 this establishes the inequality

(12) $$LM \leq LR \leq W \quad .$$

3. Which Tests Should Be Used?

Equations (9) through (11) allow us to express any one of the three tests as a function of any other. Furthermore, we can transform the W statistic into an F statistic by a standard degrees of freedom adjustment. Thus for small samples we can express any one of the three tests as a transformation of an F test for which exact critical values can be calculated.

However, few researchers go through all that trouble. Most commonly, the value of the test statistic is simply compared to the critical value c obtained from a standard Chi-square table. This is the source of possible conflicts since the three tests differ numerically but are compared against the same critical value.

Let c_α be the α percent critical value of the Chi-square distribution, i.e., $Pr\{\chi^2 > c_\alpha\}\alpha$. We can now calculate the probability of a Type I error for any one of the tests. In particular we have:

$$(13) \qquad P_I(W) = Pr\{W > c_\alpha\}$$

$$(14) \qquad P_I(LM) = Pr\{LM > c_\alpha\} \quad .$$

By expressing LM as a function of W we get

$$\begin{aligned}(15) \qquad P_I(LM) &= Pr\{W \left(\frac{1}{1 + W/n}\right) > c_\alpha\} \\ &= Pr\{W(1 - c_\alpha/n) > c_\alpha\} \\ &= Pr\{W > c_\alpha \cdot [\frac{1}{1 - c_\alpha/n}]\} < P_I(W) \quad .\end{aligned}$$

We can also establish that $P_I(W) > \alpha$:

$$(16) \qquad P_I(W) = \Pr\{W > c_\alpha\}$$

$$= \Pr\{F \cdot q(\frac{1}{1 - k/n}) > c_\alpha\}$$

$$= \Pr\{F > \frac{c_\alpha}{q}(1 - k/n)\} > \alpha \quad .$$

Since c_α/q is less than or equal to the critical value of an F distribution for the same level of significance, and $(1 - k/n) < 1$, $P_I(W)$ must be larger than α. By combining Eqs. (15) and (16) we can compare $P_I(LM)$ to α.

$$(17) \qquad P_I(LM) = \Pr\{W > c_\alpha[\frac{1}{1 - c_\alpha/n}]\}$$

$$= \Pr\{F \cdot q(\frac{1}{1 - k/n}) > c_\alpha[\frac{1}{1 - c_\alpha/n}]\}$$

$$= \Pr\{F > \frac{c_\alpha}{q}[\frac{1 - k/n}{1 - c_\alpha/n}]\} \quad .$$

If the bracketed expression in Eq. (17) is less than or equal to one, $P_I(LM)$ is larger than α. In other words, if $c_\alpha \geq k$, we have the relationship

$$(18) \qquad P_I(W) > P_I(LM) > \alpha \qquad | \; c_\alpha \geq k \quad .$$

The interesting part about this relationship is that it depends not only on q and k, but also on the level of significance chosen. If we are estimating five parameters with one constraint, we get the critical values $c_{\alpha_1} = 3.841$ for $\alpha_1 = .05$, and $c_{\alpha_2} = 6.635$ for $\alpha_2 = .01$. From Eq. (18) we can determine that at significance level α_2, the LM

test is certainly the more accurate test, i.e., $P_I(LM)$ is closer to the postulated value of α_2 than $P_I(W)$. For significance level α_1 the question is open, and depends on the difference between c_α/q and the corresponding critical value for the F distribution which in turn depends on the sample size.

4. Conclusions

We have shown that the well known inequality relation between the LM, W and LR tests does not need to lead to conflicting results. Since the tests are monotonic functions of each other, as well as of the familiar F test, we can calculate the precise critical values that will equate the probability of a Type I error for all three tests. Under these circumstances, the three tests will have exactly the same probability of rejection and identical power. Conflicting results are impossible.

If we use the same critical value for the three tests, i.e., c_α obtained from the Chi-square distribution towards which all three tests converge, conflicts are possible. However, under certain circumstances we can show that $P_I(LM)$ is closer to α than $P_I(W)$, and by inference $P_I(LR)$ which is situated between $P_I(W)$ and $P_I(LM)$, which should lead us to prefer the LM test since it is more accurate, i.e., its probability of rejection is closer to the postulated value α.

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